Spectral analysis and an area-preserving extension of a piecewise linear intermittent map

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We investigate the spectral properties of a one-dimensional piecewise linear intermittent map, which has not only a marginal fixed point but also a singular structure suppressing injections of the orbits into neighborhoods of the marginal fixed point. We explicitly derive generalized eigenvalues and eigenfunctions of the Frobenius-Perron operator of the map for classes of observables and piecewise constant initial densities, and it is found that the Frobenius-Perron operator has two simple real eigenvalues 1 and $\lambda_d \in (-1,0)$ and a continuous spectrum on the real line [0,1]. From these spectral properties, we also found that this system exhibits a power law decay of correlations. This analytical result is found to be in a good agreement with numerical simulations. Moreover, the system can be extended to an area-preserving invertible map defined on the unit square. This extended system is similar to the baker transformation, but does not satisfy hyperbolicity. A relation between this area-preserving map and a billiard system is also discussed.

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I. INTRODUCTION

In the past decades, a lot of studies have been devoted to investigations of the relations between microscopic chaos and nonequilibrium behaviors such as relaxation and transport, and it has been found that microscopic chaos plays essential roles in nonequilibrium processes [1,2]. For example, it is well known that, for fully chaotic (hyperbolic) systems, correlation functions decay exponentially and their decay rates are characterized by the discrete eigenvalues of its Frobenius-Perron (FP) operator [3–7]. As one of the examples of hyperbolic systems that permit detailed calculations, the baker transformation has been studied extensively and its spectral properties of FP operator are fully understood [8]. In addition, the baker map is considered as an abstract model of chaotic Hamiltonian systems, because it has the area-preserving property, which is a universal feature of the Poincaré map of Hamiltonian systems of two degrees of freedom [9]. In fact, similarities of the baker map to the Lorentz gas with finite horizon have been pointed out [10].

In contrast to hyperbolic systems, the dynamics in generic Hamiltonian systems is more complicated and diverse. When the phase space of a Hamiltonian system consists of integrable (torus) and nonintegrable components (chaos), the power law decay of correlations is frequently observed [11–16]. Although such kinds of systems—i.e., systems with mixed-type phase spaces—are more generic than integrable or fully chaotic systems, a theoretical understanding of their statistical properties is not enough; for example, the ergodic and mixing properties of chaotic components of generic systems are still unclear from the theoretical point of view.

For understanding the subexponential decay of correlation functions in dynamical systems, nonhyperbolic one-dimensional maps have been studied by several authors [17–26] and they have found power law decays of correlations in their models. Therefore, it is natural to imagine a close connection of these nonhyperbolic maps and mixed-type Hamiltonian systems; however, extensions of these maps to two-dimensional area-preserving systems are unknown. Thus, in this paper, we introduce a modified version of the one-dimensional intermittent map studied in Ref. [19] and extend it to an area-preserving system. This area-preserving map, which is similar to the baker transformation, may be considered as an abstract model of mixed-type Hamiltonian systems.

Our theoretical treatment is mainly based on Ref. [19], where a piecewise linear version of the Pomeau-Manneville map [27,28] is proposed and its generalized spectral properties of the FP operator in a sense of Refs. [29,30] have been elucidated. Their model has a marginal fixed point and exhibits power law decays of correlations which they have found to be the outcome of a continuous spectrum of the FP operator. In addition to a marginal fixed point, the piecewise linear map studied in the present paper has a singular structure, which suppresses injections of the orbits into neighborhoods of the marginal fixed point. Due to this property, the uniform density is invariant under time evolution and the map can be extended to an area-preserving map on the unit square. And it is shown that generalized eigenvalues of the FP operator consists of two simple real eigenvalues 1 and $\lambda_d \in (-1,0)$ and a continuous spectrum on the real interval [0,1]. It is also shown that correlation functions exhibit a power law decay due to the continuous spectrum.

This paper is organized as follows. In Sec. II, we introduce the piecewise linear map and define the FP operator, observables, and initial densities. In Sec. III, we derive the spectral decomposition of the FP operator. In Sec. IV, long-time behaviors are analyzed and some numerical results are displayed. We also discuss the extension of our model to an area-preserving invertible map in Sec. IV. Section V is devoted to summary and remarks that include comments on differences between our model and the one in Ref. [19] and about similarities to a billiard system.

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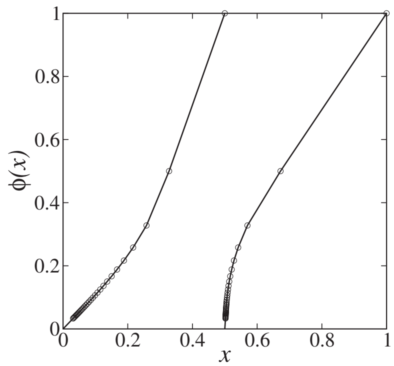

FIG. 1. The piecewise linear map $\phi(x)$ (solid line) for $b=0.5$ and $\beta=1.3$. The circles indicate end points of the straight line segments.

II. PIECEWISE LINEAR MAP

A. Definition of the map

The dynamical system we consider in this paper is the piecewise linear map shown in Fig. 1. This map $\phi(x):[0,1]\to[0,1]$ consists of two parts: the left $0\leqslant x<b$ and the right part $b\leqslant x<1$. Each part is formed by an infinite number of straight line segments; the circles plotted in Fig. 1 indicate end points of these segments. The map $\phi(x)$ is defined by

$$\phi(x)=\begin{cases}\eta_k^-(x-\xi_k^-)+\xi_{k-1}^- & \text{for } x\in[\xi_k^-,\xi_{k-1}^-)\ (k=1,2,\ldots),\\ \eta_k^+(x-\xi_k^+)+\xi_{k-1}^- & \text{for } x\in[\xi_k^+,\xi_{k-1}^+)\ (k=1,2,\ldots).\end{cases} \tag{1}$$

In this definition, ξ_k^- $(k=0,1,\ldots)$ represent the horizontal coordinates of the end points of the segments on the left part $(x<b)$ and they are defined as

$$\xi_0^-=b,$$

$$\xi_{k-1}^- - \xi_k^- = \frac{b}{\zeta(\beta)}\left(\frac{1}{k}\right)^{\beta} \quad \text{for } k=1,2,\ldots, \tag{2}$$

where $\beta>1$ is a parameter and $\zeta(\beta)=\sum_{n=1}^{\infty}1/n^{\beta}$ is the Riemann zeta function. And η_k^- is the slope of the kth segment, defined as

$$\eta_k^-=\frac{\xi_{k-2}^- - \xi_{k-1}^-}{\xi_{k-1}^- - \xi_k^-} \tag{3}$$

$$=\begin{cases}\dfrac{1-b}{b}\zeta(\beta) & \text{for } k=1,\\ \left(\dfrac{k}{k-1}\right)^{\beta} & \text{for } k=2,3,\ldots,\end{cases} \tag{4}$$

where we define $\xi_{-1}^-=1$ for convenience. This definition of the left part $(0<x<b)$ is the same as that of the piecewise linear Pomeau-Manneville map proposed in Ref. [19]. The map $\phi(x)$ can be approximated as $\phi(x)\sim x+Cx^{\beta/(\beta-1)}$ when $x\to 0+$, where C is a constant. Thus the origin $x=0$ is a marginal fixed point.

In the same way, ξ_k^+ $(k=0,1,\ldots)$ are the horizontal coordinates of the end points of the segments on the right part and defined as

$$\xi_1^+=b\left(1+\frac{1}{\zeta(\beta)}\right), \quad \xi_0^+=1,$$

$$\xi_{k-1}^+ - \xi_k^+ = \frac{b}{\zeta(\beta)}\left(\frac{1}{(k-1)^{\beta}}-\frac{1}{k^{\beta}}\right) \quad \text{for } k=2,3,\ldots, \tag{5}$$

and η_k^+ is the slope of the k-th segment and defined as

$$\eta_k^+=\frac{\xi_{k-2}^- - \xi_{k-1}^-}{\xi_{k-1}^+ - \xi_k^+} \tag{6}$$

$$=\begin{cases}\dfrac{1-b}{1-b[1+1/\zeta(\beta)]} & \text{for } k=1,\\ \dfrac{k^{\beta}}{k^{\beta}-(k-1)^{\beta}} & \text{for } k=2,3,\ldots\ .\end{cases} \tag{7}$$

This is the definition of the right part of the map $\phi(x)$, $b\leqslant x<1$. This part is different from Pomeau-Manneville-type maps and is similar to a map proposed by Artuso and Cristadoro [18]. $\phi(x)$ behaves as $\phi(x)\sim(x-b)^{(\beta-1)/\beta}$ when $x\to b+$. Therefore the derivative $\phi'(x)$ of the map is divergent at $x=b+$.

We assume that $\eta_1^+>0$—i.e.,

$$b<\frac{\zeta(\beta)}{1+\zeta(\beta)}. \tag{8}$$

Note that the uniform density on the interval $[0,1]$ is invariant under the time evolution of this map because the relation $1/\eta_k^+ + 1/\eta_k^- = 1$ is satisfied for $k=1,2,\ldots$. Figure 1 shows the shape of the map $\phi(x)$ for $\beta=1.3$ and $b=0.5$. There is a singular structure near $x=b$, which suppresses injections of the orbits into neighborhoods of the marginal fixed point $x=0$. This system can be easily extended to a two-dimensional area-preserving map, whose dynamics of the expanding direction is given by the map $\phi(x)$. This will be discussed in Sec. IV.

B. FP operator on functional spaces

The FP operator $\hat{P}$ and its adjoint $\hat{P}^*$ are defined by

$$\hat{P}\rho(x)=\int_0^1 dy\,\delta(x-\phi(y))\rho(y), \tag{9}$$

$$\hat{P}^*A(x)=A(\phi(x)), \tag{10}$$

respectively. We also define an inner product (A,ρ) as the average of an observable $A(x)$ with respect to a density $\rho(x)$,

$$(A,\rho)=\int_0^1 dx A(x)\rho(x). \tag{11}$$

Then, the average of $A(x)$ at time t with respect to an initial density $\rho(x)$ is given by $(A,\hat{P}^t\rho)=(\hat{P}^{*t}A,\rho)$.

Let us consider that an observable $A(x)$ such as the inequality

$$|A(x)-a_0-a_1x| \leqslant Kx^{\beta/(\beta-1)} \tag{12}$$

holds for some positive constant K, where the constants a_0 and a_1 satisfy $a_0=A(0)$ and $a_1=A'(0)$, respectively [19]. This function is bounded on [0,1] and smooth near the origin. And we define a set X_O as the functional space which consists of such observables. This functional space is invariant under the action of the adjoint of the FP operator $\hat{P}^*$—namely, $\hat{P}^*A(x)\in X_O$ if $A(x)\in X_O$. If we define the norm as

$$\|A(x)\|_O = |a_0| + |a_1| + \sup_x |A(x)| + \sup_x \frac{|A(x)-a_0-a_1x|}{x^{\beta/(\beta-1)}}, \tag{13}$$

then this functional space becomes a Banach space with respect to this norm. Note that this space is dense in the Hilbert space $L^2[0,1]$ of the square-integrable functions on [0,1].

Furthermore, we restrict initial densities to be piecewise constant [24],

$$\rho(x) = \tilde{\rho}_k \text{ if } x\in[\xi_k^-,\xi_{k-1}^-) \ (k=0,1,2,\ldots). \tag{14}$$

We also assume the following properties for $k=1,2,\ldots$:

$$\tilde{\rho}_k = \sum_{l=0}^{\infty} \rho_l \left(\frac{k}{k+l}\right)^{\beta} \text{ with } \sum_{l=0}^{\infty} |\rho_l|\theta^l < +\infty, \tag{15}$$

where $\theta>1$ is a constant. We also assume the normalization condition

$$\frac{b}{\zeta(\beta)}\sum_{k=1}^{\infty}\frac{\tilde{\rho}_k}{k^{\beta}} + (1-b)\tilde{\rho}_0 = 1. \tag{16}$$

In the following sections, we use this condition [Eq. (16)] only for clarity of exposition. But it is not essential and almost the same result can be obtained without it.

We define a set X_D as the functional space which consists of such densities. This functional space is invariant under the action of the FP operator; namely, $\hat{P}\rho\in X_D$ if $\rho\in X_D$. And if the norm of this space is defined as

$$\|\rho\|_D = \sum_{l=0}^{\infty}|\rho_l|\theta^l, \tag{17}$$

this functional space becomes a Banach space. The functional space X_D is not dense in $L^2[0,1]$.

In the above definition for initial densities, we assume that the initial densities are constant on the interval $[b,1)$—i.e., the right part of the map. Although it seems to be a strong restriction and it is possible to extend to the densities piecewise constant also in $[\xi_k^+,\xi_{k-1}^+)$, this extension does not make any changes for the long-time behaviors because for such densities we have $\hat{P}\rho(x)\in X_D$.

III. SPECTRAL ANALYSIS

The purpose of this section is to derive a spectral decomposition of the average $(A,\hat{P}^t\rho)$. First, we derive the matrix elements of the resolvent operator of $\hat{P}$; then, the average $(A,\hat{P}^t\rho)$ is obtained by an integral transformation of the matrix elements of the resolvent. Finally, deforming the integration path, we have the spectral decomposition.

A. Matrix elements of the resolvent operator

Let us define the matrix elements of the resolvent of the FP operator $\hat{P}$ as

$$\left(A,\frac{1}{z-\hat{P}}\rho\right) = \sum_{t=0}^{\infty}\frac{1}{z^{t+1}}(A,\hat{P}^t\rho) = \sum_{k=1}^{\infty}[\tilde{\rho}_k\hat{B}_k^-(z)+\tilde{\rho}_0\hat{B}_k^+(z)], \tag{18}$$

where $\hat{B}_k^{\pm}(z)$ are defined below [see Eq. (21)]. Let us rewrite $(A,\hat{P}^t\rho)$ as

$$(A,\hat{P}^t\rho) = \int_0^1 dxA(\phi^t(x))\rho(x) = \sum_{k=1}^{\infty}[\tilde{\rho}_kB_k^-(t)+\tilde{\rho}_0B_k^+(t)], \tag{19}$$

where $B_k^{\pm}(t)$ are defined for $k=1,2,\ldots$ by

$$B_k^{\pm}(t) \equiv \int_{\xi_k^{\pm}}^{\xi_{k-1}^{\pm}} dxA(\phi^t(x)). \tag{20}$$

Then $\hat{B}_k^{\pm}(z)$ are defined for $k=1,2,\ldots$ by

$$\hat{B}_k^{\pm}(z) \equiv \sum_{t=0}^{\infty}\frac{B_k^{\pm}(t)}{z^{t+1}}. \tag{21}$$

From Eq. (20), we have the following recursion relations for $B_k^{\pm}(t)$:

$$B_1^{\pm}(t+1) = \frac{1}{\eta_1^{\pm}}\sum_{k=1}^{\infty}B_k^+(t), \tag{22}$$

$$B_k^{\pm}(t+1) = \frac{1}{\eta_k^{\pm}}B_{k-1}^-(t) \text{ for } k=2,3,\ldots\ . \tag{23}$$

From Eqs. (21)–(23), we have the recursion relations for $\hat{B}_k^{\pm}(z)$:

$$\hat{B}_1^{\pm}(z) = \frac{B_1^{\pm}(0)}{z} + \frac{1}{z\eta_1^{\pm}}\sum_{k=1}^{\infty}\hat{B}_k^+(z), \tag{24}$$

$$\hat{B}_k^{\pm}(z) = \frac{B_k^{\pm}(0)}{z} + \frac{1}{z\eta_k^{\pm}}\hat{B}_{k-1}^-(z) \text{ for } k=2,3,\ldots\ . \tag{25}$$

Using these relations recursively, the following equation can be derived for $k=1,2,\ldots$:

$$\hat{B}_k^-(z) = \sum_{m=0}^{k-1}\frac{B_{k-m}^-(0)}{z^{m+1}}\left(\frac{k-m}{k}\right)^{\beta} + \frac{1}{z^kk^{\beta}}\frac{b}{(1-b)\zeta(\beta)}\sum_{m=1}^{\infty}\hat{B}_m^+(z). \tag{26}$$

Furthermore, from Eqs. (25) and (26), we obtain, for $k=2,3,\ldots$,

$$\hat{B}_k^+(z) = \frac{B_k^+(0)}{z} + \frac{1}{\eta_k^+}\left[\sum_{m=0}^{k-2}\frac{B_{k-m-1}^-(0)}{z^{m+2}}\left(\frac{k-m-1}{k-1}\right)^\beta\right. \tag{27}$$

$$\left. + \frac{1}{z^k(k-1)^\beta}\frac{b}{(1-b)\zeta(\beta)}\sum_{m=1}^{\infty}\hat{B}_m^+(z)\right]. \tag{28}$$

Summing up Eq. (24) and Eqs. (28), we have

$$\sum_{k=1}^{\infty}\hat{B}_k^+(z) = \frac{\Phi(z)}{Z(z)}, \tag{29}$$

where $Z(z)$ and $\Phi(z)$ are defined by

$$Z(z) \equiv z - \frac{1}{\eta_1^+} - \frac{b}{(1-b)\zeta(\beta)}\sum_{k=1}^{\infty}\frac{1}{\eta_{k+1}^+ z^k k^\beta}, \tag{30}$$

$$\Phi(z) \equiv \sum_{k=1}^{\infty}B_k^+(0) + \sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\frac{B_l^-(0)}{\eta_{m+l+1}^+ z^{m+1}}\left(\frac{l}{m+l}\right)^\beta. \tag{31}$$

The two z^{-1} power series on the right-hand sides of Eqs. (30) and (31) are absolutely convergent for $|z|>1$; hence, the functions $Z(z)$ and $\Phi(z)$ are analytic there. From Eqs. (26) and (29)–(31), the matrix elements of the resolvent [Eq. (18)] can be rewritten as

$$\begin{aligned}\left(A, \frac{1}{z-P}\rho\right) &= \frac{\Psi(z)\Phi(z)}{Z(z)} + \sum_{l=1}^{\infty}\sum_{k=l}^{\infty}\frac{\tilde{\rho}_k B_l^-(0)}{z^{k-l+1}}\left(\frac{l}{k}\right)^\beta \\ &= \frac{\Psi(z)\Phi(z)}{Z(z)} + \frac{(1-b)\zeta(\beta)}{b}[\Xi(z) - B_1^-(0)\tilde{\rho}_0],\end{aligned} \tag{32}$$

where we define the functions $\Psi(z)$ and $\Xi(z)$ as

$$\Psi(z) \equiv \tilde{\rho}_0 + \frac{b}{(1-b)\zeta(\beta)}\sum_{k=1}^{\infty}\frac{\tilde{\rho}_k}{z^k k^\beta} \tag{33}$$

and

$$\begin{aligned}\Xi(z) = B_1^-(0)\Psi(z) + \sum_{l=2}^{\infty}B_l^-(0)l^\beta z^{l-1}\Bigg[\Psi(z) - \tilde{\rho}_0 \\ - \frac{b}{(1-b)\zeta(\beta)}\sum_{k=1}^{l-1}\frac{\tilde{\rho}_k}{z^k k^\beta}\Bigg],\end{aligned} \tag{34}$$

respectively. $\Psi(z)$ and $\Xi(z)$ are also analytic for $|z|>1$, as each representing series in Eqs. (33) and (34) is absolutely convergent there.

B. Analytic properties of individual functions

For $r\equiv|z|>1$, we obtain the average (A,ρ_t) by an integral transformation of the resolvent [Eq. (32)] as follows:

$$\begin{aligned}(A,\hat{P}^t\rho) &= \oint_{|z|=r}\frac{dz}{2\pi i}z^t\left(A, \frac{1}{z-P}\rho\right) \\ &= \oint_{|z|=r}\frac{dz}{2\pi i}z^t\left[\frac{\Psi(z)\Phi(z)}{Z(z)} + \frac{(1-b)\zeta(\beta)}{b}\Xi(z)\right],\end{aligned} \tag{35}$$

where the integration path is taken in a counterclockwise direction. In order to deform this integration path into a unit disk $|z|<1$ and derive the spectral decomposition, we study the analytic properties of the functions in the integrand of Eq. (35) in this subsection. With the help of the identity

$$\frac{1}{k^\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty ds s^{\beta-1}e^{-ks}, \tag{36}$$

we have analytic continuations of the functions $Z(z)$, $\Phi(z)$, and $\Psi(z)$ into the unit disk $|z|<1$. For example, $\Phi(z)$ can be analytically continued as follows:

$$\begin{aligned}\Phi(z) = \sum_{k=1}^{\infty}B_k^+(0) + \frac{1}{\Gamma(\beta)}\sum_{l=1}^{\infty}\frac{B_l^-(0)l^\beta}{z} \\ \times\int_0^\infty ds s^{\beta-1}e^{-ls}(1-e^{-s})\sum_{m=0}^{\infty}(z^{-1}e^{-s})^m\end{aligned} \tag{37}$$

$$= \sum_{k=1}^{\infty}B_k^+(0) + \frac{1}{\Gamma(\beta)}\sum_{l=1}^{\infty}B_l^-(0)l^\beta\int_0^\infty ds\frac{s^{\beta-1}e^{-ls}}{z-e^{-s}}(1 - e^{-s}), \tag{38}$$

where these calculations are justified because the convergence of the summation is uniform in s for $|z|>1$. Note that each integral on the right-hand side (RHS) of Eq. (38) is analytic except for the real interval [0, 1]. Then the function $\Phi(z)$ expressed as Eq. (38) is analytic except for the cut [0, 1], because the second infinite sum in Eq. (38) is absolutely convergent—i.e.,

$$\frac{1}{\Gamma(\beta)}\sum_{l=1}^{\infty}\left|B_l^-(0)l^\beta\int_0^\infty ds\frac{s^{\beta-1}e^{-ls}}{z-e^{-s}}(1-e^{-s})\right| \leqslant \frac{b\sup_x|A(x)|}{\zeta(\beta)d(z,[0,1])}, \tag{39}$$

where $d(z,[0,1])$ is the distance between z and the real interval [0, 1].

In the same way, we obtain analytical continuations of the functions $Z(z)$ and $\Psi(z)$:

$$Z(z) = (z-1)\left[1 + \frac{b}{(1-b)\zeta(\beta)\Gamma(\beta)}\int_0^\infty ds\frac{s^{\beta-1}e^{-s}}{z-e^{-s}}\right], \tag{40}$$

$$\Psi(z) = \tilde{\rho}_0 + \frac{b}{(1-b)\zeta(\beta)\Gamma(\beta)}\sum_{l=0}^{\infty}\rho_l\int_0^{\infty} ds \frac{s^{\beta-1}e^{-(l+1)s}}{z-e^{-s}}. \tag{41}$$

These expressions for the functions $Z(z)$ and $\Psi(z)$ are also analytic except for the cut [0, 1]. And the function $\Xi(z)$ can be also analytically continued into the unit disk $|z|<1$ in the same way. Here, however, we rewrite the function $\Xi(z)$ in the following form:

$$\Xi(z) = \Psi(z)\sum_{l=1}^{\infty} l^{\beta}B_l^{-}(0)z^{l-1} - \sum_{l=2}^{\infty} l^{\beta}B_l^{-}(0)z^{l-1}\left[\tilde{\rho}_0 + \frac{b}{(1-b)\zeta(\beta)}\sum_{k=1}^{l-1}\frac{\tilde{\rho}_k}{z^k k^{\beta}}\right]. \tag{42}$$

The infinite sums in this expression of the function $\Xi(z)$ are absolutely convergent for $|z|<1$.

Let us consider the zeros of the function $Z(z)$. First we define a function $\Omega(z)$ as $\Omega(z)\equiv Z(z)/(z-1)$. And if $\mathrm{Im}(z)\neq 0$, then

$$\mathrm{Im}\,\Omega(z) = -\frac{b\,\mathrm{Im}(z)}{(1-b)\zeta(\beta)\Gamma(\beta)}\int_0^{\infty} ds\frac{s^{\beta-1}e^{-s}}{|z-e^{-s}|^2} \neq 0. \tag{43}$$

And also if $\mathrm{Im}(z)=0$ and $\mathrm{Re}(z)>1$, then $\Omega(z)>0$, because $z-e^{-s}>0$. Thus there is no zero of $Z(z)$ in these regions.

Next if $\mathrm{Im}(z)=0$ and $\mathrm{Re}(z)<0$, we have

$$\Omega'(z) = -\frac{b}{(1-b)\zeta(\beta)\Gamma(\beta)}\int_0^{\infty} ds\frac{s^{\beta-1}e^{-s}}{(z-e^{-s})^2} < 0 \tag{44}$$

and

$$\begin{aligned}\Omega(-1) &= 1 - \frac{b}{(1-b)\zeta(\beta)\Gamma(\beta)}\int_0^{\infty} ds\frac{s^{\beta-1}}{e^{s}+1} \\ &> 1 - \frac{b}{(1-b)\zeta(\beta)\Gamma(\beta)}\int_0^{\infty} ds\frac{s^{\beta-1}}{e^{s}} \\ &= 1 - \frac{b}{(1-b)\zeta(\beta)} > 0.\end{aligned} \tag{45}$$

The last inequality holds because of the inequality in (8), the validity of which is derived from the expression $\eta_k^{+}>0$ for $k=1$ in (7). And it is easy to see that $\Omega(z)\to -\infty$ as $z\to 0$. Therefore, on the real interval $[-1,0]$, $\Omega(z)$ has the unique zero λ_d of order 1.

C. Spectral decomposition

From the results of the last subsection, the contour of the integration of Eq. (35) can be deformed into the unit disk $|z|<1$ as

$$(A,\hat{P}^t\rho) = \lim_{\eta\to 0}\oint_{|z-\lambda_d|=\eta}\frac{dz}{2\pi i}z^t\frac{\Psi(z)\Phi(z)}{Z(z)} + \lim_{\eta\to 0}\oint_C \frac{dz}{2\pi i}z^t\left[\frac{\Psi(z)\Phi(z)}{Z(z)} + \frac{(1-b)\zeta(\beta)}{b}\Xi(z)\right], \tag{46}$$

where the integration path C is defined by

$$C \equiv \{z|\,|z|=\eta,\ |z-1|=\eta \text{ or } z=\lambda \pm i0\ (\eta<\lambda<1-\eta). \tag{47}$$

The first term on the RHS of Eq. (46) corresponds to the simple pole at $z=\lambda_d$ and thus can be calculated by the residue theorem, and Eq. (46) is rewritten as

$$(A,\hat{P}^t\rho) = \lambda_d^{\,t}\frac{\Psi(\lambda_d)\Phi(\lambda_d)}{(\lambda_d-1)\Omega'(\lambda_d)} + \lim_{\eta\to 0}\oint_C\frac{dz}{2\pi i}z^t\left[\frac{\Psi(z)\Phi(z)}{Z(z)} + \frac{(1-b)\zeta(\beta)}{b}\Xi(z)\right]. \tag{48}$$

Let us consider the integral on the RHS of Eq. (48). Because of the facts (see the Appendix)

$$\lim_{z\to 0} z^{\alpha}\Xi(z) = 0 \tag{49}$$

and

$$\lim_{z\to 0} z^{\alpha}\frac{\Phi(z)\Psi(z)}{Z(z)} = 0, \tag{50}$$

for $\forall\ \alpha>0$, the contribution from the integration around the origin, $\{|z|=\eta\}$, vanishes.

Similarly, we obtain (see the Appendix)

$$\lim_{z\to 1}(z-1)\Xi(z) = 0, \tag{51}$$

and then the contribution from the integral around $z=1$, $\{|z-1|=\eta\}$, vanishes for the second term of the integrand. On the other hand, we have

$$\begin{aligned}\lim_{\eta\to 1}\int_{|z-1|=\eta} dz z^t\frac{\Psi(z)\Phi(z)}{Z(z)} &= \lim_{z\to 1}(z-1)\frac{\Phi(z)\Psi(z)}{Z(z)} \\ &= \int_0^1 dx A(x)\left[(1-b)\tilde{\rho}_0 + \frac{b}{\zeta(\beta)}\sum_{k=1}^{\infty}\frac{\tilde{\rho}_k}{k^{\beta}}\right] = \int_0^1 dx A(x),\end{aligned} \tag{52}$$

where we have used the normalization condition (16). The RHS of Eq. (52) is the average of $A(x)$ with respect to the invariant density which is uniform for the map $\phi(x)$; therefore, Eq. (52) gives the average value of the invariant state.

Finally we have to evaluate the integral along the cut. For this purpose, we define the new functions $\hat{\Omega}(\lambda)$, $f_{\lambda}(x)$, and $\nu_l(x)$ for $0<\lambda<1$ as follows:

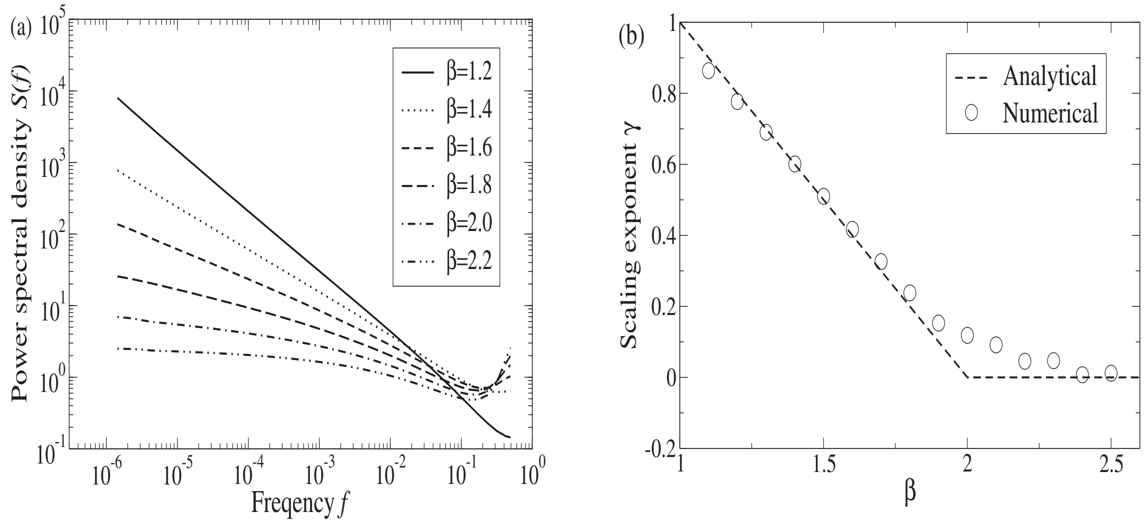

FIG. 2. (a) Power spectral densities $S(f)$ of time series $A(x(t))$ (in log-log form) for five different values of the system parameter β: $\beta=1.2$ (solid line), 1.4 (dotted line), 1.6 (dashed line), 1.8 (long-dashed line), 2.0 (dash-dotted line), and 2.2 (dashed-double-dotted line). The other parameter b is set to 1/2. These PSDs are obtained by averaging over 20 000 initial conditions uniformly distributed in [0,1]. (b) The scaling exponent γ of the PSD $S(f)\sim 1/f^{\gamma}$ as a function of β. The circles are the numerical results obtained by least-squares fitting in the low-frequency region (below $f=10^{-4}$) of the PSDs $S(f)$, and the dashed line is the theoretical prediction [Eq. (75)].

$$\hat{\Omega}(\lambda) \equiv 1 + \frac{b}{(1-b)\zeta(\beta)\Gamma(\beta)}\int_0^\infty ds\mathcal{P}\frac{s^{\beta-1}e^{-s}}{\lambda - e^{-s}}, \tag{53}$$

$$f_\lambda(x) \equiv \frac{1}{1-b}\left[\chi_0(x) + \frac{1}{\Gamma(\beta)}\sum_{l=1}^{\infty} l^\beta \chi_l(x) \times \int_0^\infty ds\mathcal{P}\frac{s^{\beta-1}e^{-ls}}{\lambda - e^{-s}}(1-e^{-s})\right], \tag{54}$$

$$\nu_l(\lambda) \equiv \frac{1}{\zeta(\beta)\Gamma(\beta)}\int_0^\infty ds\mathcal{P}\frac{s^{\beta-1}e^{-(l+1)s}}{\lambda - e^{-s}}, \tag{55}$$

where $\mathcal{P}$ means the Cauchy's principle value and $\chi_l(x)$ is defined, for $l=0,1,\ldots,$ by

$$\chi_l(x) = \begin{cases} 1 & \text{for } \xi_l^- \leqslant x < \xi_{l-1}^-, \\ 0 & \text{otherwise.} \end{cases} \tag{56}$$

Note that we have defined $\xi_{-1}=1$ in Sec. II. These functions are related to the real parts of the functions $\{Z(z),\Phi(z),\Psi(z)\}$ near the cut: $\mathrm{Re}\,Z(\lambda\pm i0)=(\lambda-1)\hat{\Omega}(\lambda)$, $\mathrm{Re}\,\Phi(\lambda\pm i0)=(1-b)\int_0^1 dxA(x)f_\lambda(x)$, and $\mathrm{Re}\,\Psi(\lambda\pm i0)=\tilde{\rho}_0+b/(1-b)\Sigma_{l=0}^{\infty}\rho_l\nu_l(\lambda)$.

It can also be shown for the imaginary parts of the functions $\{Z(z),\Phi(z),\Psi(z)\}$ near the cut that

$$\frac{\mathrm{Im}\,\Phi(\lambda - i0)}{\mathrm{Im}\,Z(\lambda + i0)} = \frac{(1-b)\zeta(\beta)}{b}\sum_{l=1}^{\infty} l^\beta B_l^-(0)\lambda^{l-1}, \tag{57}$$

$$\frac{\mathrm{Im}\,\Psi(\lambda - i0)}{\mathrm{Im}\,Z(\lambda + i0)} = \frac{1}{1-\lambda}\sum_{l=0}^{\infty}\rho_l\lambda^l. \tag{58}$$

Using these functions, we can calculate the integral along the cut as follows [19]:

$$\lim_{\eta\to 0}\int_{C\backslash\{|z|=\eta\cup|z-1|=\eta\}}\frac{dz}{2\pi i}z^t\left[\frac{\Psi(z)\Phi(z)}{Z(z)} + \frac{(1-b)\zeta(\beta)}{b}\Psi(z)\sum_{l=1}^{\infty} l^\beta B_l^-(0)z^{l-1}\right] = \int_0^1 \frac{d\lambda}{\pi}\lambda^t\frac{\mathrm{Im}[Z(\lambda+i0)\Phi(\lambda-i0)]\mathrm{Im}[Z(\lambda+i0)\Psi(\lambda-i0)]}{|Z(\lambda+i0)|^2\,\mathrm{Im}\,Z(\lambda+i0)} \tag{59}$$

$$= \frac{b}{(1-b)\zeta(\beta)\Gamma(\beta)}\int_0^1 d\lambda\frac{\mathrm{Im}[Z(\lambda+i0)\Phi(\lambda-i0)]}{\mathrm{Im}\,Z(\lambda+i0)} \tag{60}$$

$$\times\frac{\lambda^t(1-\lambda)\left(\ln\frac{1}{\lambda}\right)^{\beta-1}}{|Z(\lambda+i0)|^2}\frac{\mathrm{Im}[Z(\lambda+i0)\Psi(\lambda-i0)]}{\mathrm{Im}\,Z(\lambda+i0)} \tag{61}$$

$$= \int_0^1 d\lambda(A,F_\lambda)\lambda^t(\tilde{F}_\lambda,\rho), \tag{62}$$

where we define the linear functional (A,F_λ) of the observables $A(x)$ as

$$(A,F_\lambda) \equiv N(\lambda)\int_0^1 dx\{A(x)-A(0)\}\left[f_\lambda(x) + \frac{(\lambda-1)\hat{\Omega}(\lambda)\zeta(\beta)}{b}\sum_{l=1}^{\infty} l^\beta\lambda^{l-1}\chi_l(x)\right]. \tag{63}$$

The function $N(\lambda)$ in Eq. (63) is given by

$$N(\lambda) \equiv \frac{\frac{b}{\zeta(\beta)\Gamma(\beta)(1-b)}\left(\ln\frac{1}{\lambda}\right)^{\beta-1}}{(1-\lambda)\left\{\hat{\Omega}^2(\lambda)+\left[\frac{b}{\zeta(\beta)\Gamma(\beta)(1-b)}\left(\ln\frac{1}{\lambda}\right)^{\beta-1}\right]^2\right\}}. \tag{64}$$

We also define in Eq. (62) the linear functional $(\tilde{F}_\lambda,\rho)$ of the initial densities $\rho(x)$ as

$$(\tilde{F}_\lambda,\rho) \equiv (1-b)\tilde{\rho}_0 + b\sum_{l=0}^{\infty}\rho_l\left[\nu_l(\lambda)-\frac{1-b}{b}\hat{\Omega}(\lambda)\lambda^l\right]. \tag{65}$$

Similarly, the linear functionals associated with the eigenvalues $z=\lambda_d$ and $z=1$, which have been obtained in Eqs. (48) and (53), can be expressed as

$$(A,F_d) \equiv \frac{1}{(\lambda_d-1)\Omega'(\lambda_d)}\int_0^1 dxA(x)f_{\lambda_d}(x), \tag{66}$$

$$(\tilde{F}_d,\rho) \equiv (1-b)\tilde{\rho}_0 + b\sum_{l=0}^{\infty}\rho_l\nu_l(\lambda_d) \tag{67}$$

and

$$(A,F_{\text{in}}) \equiv \int_0^1 dxA(x), \tag{68}$$

$$(\tilde{F}_{\text{in}},\rho) \equiv 1, \tag{69}$$

respectively. It is easy to check that $(A,F_\lambda)=(A,F_d)=0$ if $A(x)$ is a constant function and that $(\tilde{F}_\lambda,\rho)=(\tilde{F}_d,\rho)=0$ if $\rho(x)\equiv 1$.

Using these linear functionals, we can spectrally decompose the average (A,ρ_t) [Eq. (48)] at time $t=0,1,2,\ldots$ as

$$(A,\hat{P}^t\rho) = (A,F_{\text{in}})(\tilde{F}_{\text{in}},\rho) + \lambda_d^t(A,F_d)(\tilde{F}_d,\rho) + \int_0^1 d\lambda\lambda^t(A,F_\lambda)(\tilde{F}_\lambda,\rho). \tag{70}$$

The left eigenfunctions $\{F_{\text{in}},F_{\text{d}},F_\lambda\}$ satisfy the relations

$$(A,\hat{P}F_{\text{in}}) \equiv (\hat{P}^*A,F_{\text{in}}) = (A,F_{\text{in}}),$$

$$(A,\hat{P}F_d) \equiv (\hat{P}^*A,F_d) = \lambda_d(A,F_d),$$

$$(A,\hat{P}F_\lambda) \equiv (\hat{P}^*A,F_\lambda) = \lambda(A,F_\lambda). \tag{71}$$

Therefore $\{F_{\text{in}},F_{\text{d}},F_\lambda\}$ are eigenfunctions of the FP operator in a generalized sense [19,24,29,30]. On the other hand, the right eigenfunctions $\{\tilde{F}_{\text{in}},\tilde{F}_{\text{d}},\tilde{F}_\lambda\}$ satisfy the relations

$$(\hat{P}^*\tilde{F}_{\text{in}},\rho) \equiv (\tilde{F}_{\text{in}},\hat{P}\rho) = (\tilde{F}_{\text{in}},\rho),$$

$$(\hat{P}^*\tilde{F}_{\text{d}},\rho) \equiv (\tilde{F}_{\text{d}},\hat{P}\rho) = \lambda_{\text{d}}(\tilde{F}_d,\rho),$$

$$(\hat{P}^*\tilde{F}_\lambda,\rho) \equiv (\tilde{F}_\lambda,\hat{P}\rho) = \lambda(\tilde{F}_\lambda,\rho). \tag{72}$$

Therefore $\{\tilde{F}_{\text{in}},\tilde{F}_{\text{d}},\tilde{F}_\lambda\}$ are eigenfunctions of the adjoint of the FP operator in a generalized sense.

Thus we obtain the spectral decomposition, Eq. (70), where the spectrum consists of two discrete eigenvalues 1 and λ_d, and a continuous spectrum on [0, 1]. And Eq. (70) for $t=0$ shows that these eigenfunctions are complete.

IV. LONG-TIME BEHAVIORS AND AN AREA-PRESERVING EXTENSION

A. Long-time behaviors

In the previous section, we derive the spectral decomposition of the average $(A,\hat{P}^t\rho)$ and found that there is a continuous spectrum. Since long-time behaviors are controlled by eigenvalues whose absolute values are close to 1, we consider the limit $\lambda\to 1$ for the continuous spectrum. The eigenstate associated with the eigenvalue λ_d does not contribute to long-time behaviors, because this state decays exponentially. Under the assumption (12), the leading term of the left eigenstate (A,F_λ) when $\lambda\simeq 1$ is given by

$$(A,F_\lambda) \simeq K(1-\lambda)^{\beta-2}\int_0^1 dx[A(x)-A(0)], \tag{73}$$

where K is a constant. Similarly, we obtain the leading term for the right eigenstate $(\tilde{F}_\lambda,\rho)$,

$$(\tilde{F}_\lambda,\rho) \simeq (\tilde{F}_{\text{in}},\rho) - \sum_{l=0}^{\infty}\rho_l = 1-\sum_{l=0}^{\infty}\rho_l,$$

as $\lambda\simeq 1$. Using these facts and Eq. (70), we have for $t\to\infty$

$$(A,\hat{P}^t\rho) \simeq \int_0^1 dxA(x) + \frac{K'}{t^{\beta-1}}\left(1-\sum_{l=0}^{\infty}\rho_l\right)\int_0^1 dx[A(x)-A(0)], \tag{74}$$

where K' is a constant. Equation (74) shows that the correlation functions decay algebraically. From Eq. (74), it is found that the power spectral density (PSD) $S(f)$ behaves as [25]

$$S(f) \sim \begin{cases} \dfrac{1}{f^{2-\beta}} & \text{for } 1<\beta<2, \\ |\ln f| & \text{for } \beta=2, \\ \text{const} & \text{for } \beta>2. \end{cases} \tag{75}$$

Figure 2(a) shows PSDs $S(f)$ of time series $A(x(t))$, where the observable $A(x)$ is a step function defined on $x\in[0,1)$ as

$$A(x) = \begin{cases} -1 & \text{for } x\in[0,1/2), \\ 1 & \text{for } x\in[1/2,1). \end{cases} \tag{76}$$

And $x(t)$ is produced by the successive mappings of $\phi(x):x(t+1)=\phi(x(t))$. All calculations are performed in long double precision. Each PSD is obtained by averaging over 20 000 initial conditions uniformly distributed in [0, 1].

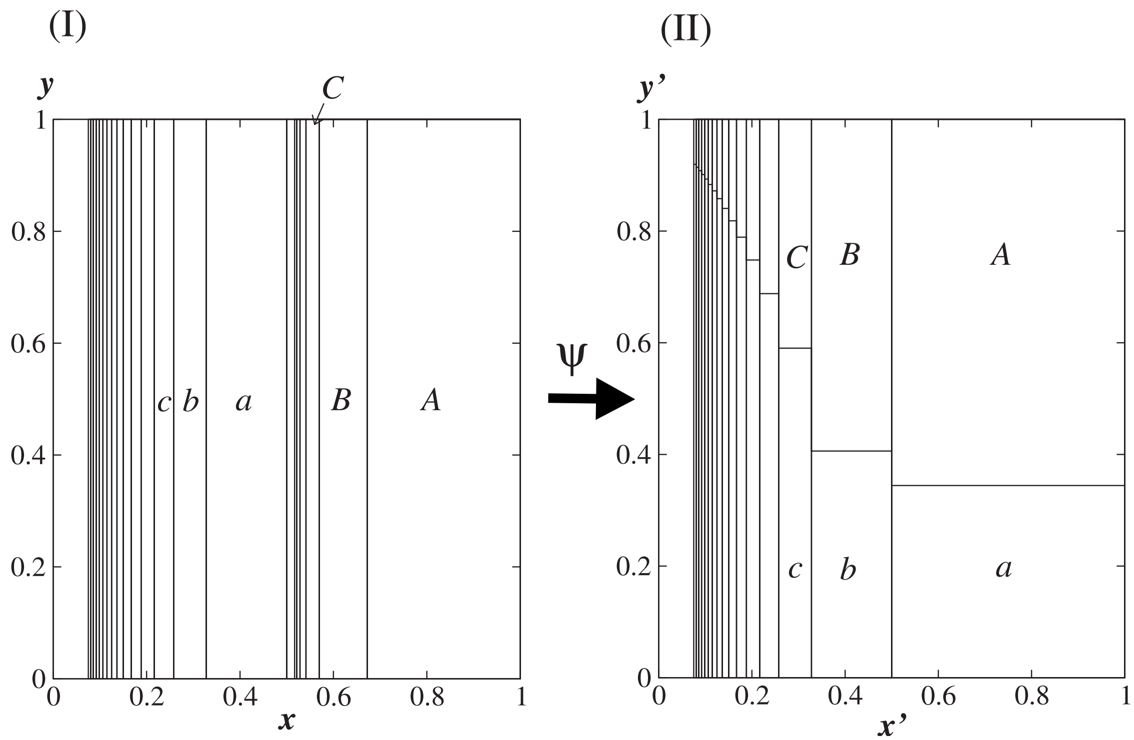

FIG. 3. The area-preserving map $\psi(x,y)$ defined by Eq. (77) for $b=0.5$ and $\beta=1.3$. The domains with labels $\{A,B,C,a,b,c\}$ in the left cell (I) are mapped into the domains with the same labels in the right (II), respectively. Each domain is uniformly stretched in the horizontal direction and uniformly squeezed in the vertical direction. There are an infinite number of such domains, and each one is mapped in a similar way. The partition is displayed only for the region $\{x \in [\xi_{15}^-, \xi_0^-]\}$ and $\{x \in [\xi_6^+, \xi_0^+]\}$ in the left cell (I), only for $x \in [\xi_{15}^-, \xi_0^+]$ in the right (II). The other regions are not displayed because the structures are too fine to see.

In Fig. 2(a), the PSDs for $\beta < 2$ exhibit clear $1/f^\gamma$ scalings in low-frequency regions. Figure 2 displays this scaling exponent γ of the PSD $S(f) \sim 1/f^\gamma$ as a function of the system parameter β. And we show the theoretical prediction derived in the above [Eq. (75)] by the dashed line in Fig. 2(b). Obviously, the numerical results show good agreement with the theoretical prediction.

B. Area-preserving extension

As stated in the introduction, the one-dimensional map investigated in the present paper can be extended to an area-preserving two-dimensional transformation defined on the unit square; this extension $\psi:[0,1)^2 \to [0,1)^2$ is defined as

$$\psi(x,y) = \begin{cases} \left(\eta_k^-(x-\xi_k^-) + \xi_{k-1}^-, \dfrac{y}{\eta_k^-}\right) & \text{for } x \in [\xi_k^-, \xi_{k-1}^-), \\ \left(\eta_k^+(x-\xi_k^+) + \xi_{k-1}^-, \dfrac{y}{\eta_k^+} + \dfrac{1}{\eta_k^-}\right) & \text{for } x \in [\xi_k^+, \xi_{k-1}^+), \end{cases} \tag{77}$$

where $k=1,2,\ldots$. Obviously, the transformation for the horizontal coordinate x is the same as the map $\phi(x)$ defined by Eq. (1) and does not depend on the vertical coordinate y. This relation between the one-dimensional map $\phi(x)$ and its area-preserving extension $\psi(x,y)$ is the same as that between the Bernoulli and baker transformations. Note that the map $\psi(x,y)$ is area preserving because the Jacobian of this map is equal to 1 everywhere. The phase space (i.e., the unit square) of this area-preserving map can be partitioned into infinite pieces like

$$[0,1)^2 = \bigcup_{k=1}^{\infty} \{(x,y) | x \in [\xi_k^-, \xi_{k-1}^-), y \in [0,1)\} \cup \bigcup_{k=1}^{\infty} \{(x,y) | x \in [\xi_k^+, \xi_{k-1}^+), y \in [0,1)\}. \tag{78}$$

See Fig. 3(I). Each piece of this partition $\{(x,y) | x \in [\xi_k^\pm, \xi_{k-1}^\pm), y \in [0,1)\}$ is uniformly stretched in the horizontal direction and uniformly squeezed in the vertical direction. The left pieces $\{(x,y) | x \in [\xi_k^-, \xi_{k-1}^-), y \in [0,1)\}$ are mapped to the bottom part of the unit cell and the right pieces $\{(x,y) | x \in [\xi_k^+, \xi_{k-1}^+), y \in [0,1)\}$ to the upper [Fig. 3(II)].

Note that although our theoretical and numerical results are for the one-dimensional map $\phi(x)$, these results are also true for this area-preserving map if an observable does not depend on the vertical coordinate y—namely, $A(x,y)=A(x)$—because the horizontal coordinate x of this area-preserving map is transformed by $\phi(x)$ and independent of y. Therefore this area-preserving extension of $\phi(x)$ has also long-time correlations with a power law decay. This fact contrasts to the results for the baker transformation, which exhibits an exponential decay of correlation functions.

Here let us define some terms for later discussions. We define the kth escape domain $\mathcal{D}_k$ as

$$\mathcal{D}_k = \{(x,y) | x \in [\xi^-_{k-1}, \xi^-_{k-2}), y \in [0,1)\} \tag{79}$$

for $k=1,2,\ldots$. For the points in kth escape domain $\mathcal{D}_k$, it takes $(k-1)$ times of the mappings ψ^n to escape from the left part $x<b$ to the right $x>b$. We also define the kth injection domain $\mathcal{D}^{\text{in}}_k$ as

$$\mathcal{D}^{\text{in}}_k = \{\psi(x,y) | x \in [\xi^+_k, \xi^+_{k-1}), y \in [0,1)\} \tag{80}$$

for $k=1,2,\ldots$. The injection domains are the upper part of the unit square, which are displayed in Fig. 3(II) by the labels $\{A,B,C,\ldots\}$. We denote the areas—namely, the Lebesgue measures—of the kth escape and injection domains as S_k and S^{in}_k, respectively. These areas obey scaling laws $S_k \sim 1/k^\beta$ and $S^{\text{in}}_k \sim 1/k^{\beta+1}$. This latter power law gives the escape time distribution [31].

V. SUMMARY AND REMARKS

In this paper, we have introduced a piecewise linear map and analyzed its spectral properties. We have derived the generalized eigenfunctions and eigenvalues explicitly for classes of observables and piecewise constant initial densities. Our model is a modified version of the map analyzed in Ref. [19]. The main difference of these two models is the normalizability of invariant densities. The invariant density of the model investigated in Ref. [19] is not normalizable for a parameter region. This is a typical property of dynamical systems with marginal fixed points [19,20,26] and is caused by divergence of invariant density at marginal fixed points.

On the other hand, the uniform density is invariant for the map $\phi(x)$ discussed in the present paper; therefore, the invariant density is normalizable for any values of the system parameters, even though our system has also a marginal fixed point. This is because the present model has the mechanism suppressing injections of the orbits into neighborhoods of the marginal fixed point and this property prevents divergences of the invariant density at the marginal fixed point. As a consequence of the normalizability, the present model does not exhibit nonstationarity, which is generically observed in maps with marginal fixed points [19,20,26].

The spectral properties of the present model are similar to those of Ref. [19] in the locations of the discrete and continuous spectra. There are two simple eigenvalues 1 and $\lambda_d \in (-1,0)$; the former corresponds to the invariant eigenstate and the latter to the oscillating one. The eigenstate associated with λ_d, however, does not contribute to the long-time behaviors of the correlation functions because it decays exponentially fast. There is also the continuous spectrum on the real interval [0, 1]; this continuous spectrum leads to the power law decay of correlation functions. We have confirmed the good agreement between the theoretical prediction and the numerical result for scaling behaviors of the PSD $S(f) \sim 1/f^\gamma$.

Furthermore, the piecewise linear map $\phi(x)$ has been extended to an area-preserving invertible map on the unit square. In contrast to the baker transformation, which is hyperbolic and shows an exponential decay of correlation functions, our model is nonhyperbolic and displays the power law decay of correlations.

As is well known, mixed-type Hamiltonian systems often exhibit a power law decay of correlation functions. The area-preserving map $\psi(x,y)$ introduced in this paper may be considered as an abstract model of mixed-type Hamiltonian systems in the following sense. Instabilities of the orbits of the map $\psi(x,y)$ [Eq. (77)] are weak in neighborhoods of the line $x=0$, and the escape time from the left part $x<b$ to the right part $x>b$ diverges as $x \to 0$. In other words, the orbits stick to the line $x=0$ for long times. This property seems to be similar to the dynamics of Hamiltonian systems near torus, cantorus, and marginally unstable periodic orbits, where chaotic orbits stick for long times.

And, in fact, similar dynamics is observed in a Poincaré map of the mushroom billiard [31], which has been proposed recently as a model of mixed type systems with sharply divided phase spaces [32–36]. In Ref. [31], it is found that an infinite partition can be constructed on a Poincaré surface using escape times from neighborhoods of the outermost tori and that the area of the escape domains and the injection domains obey the scaling relations $\mathcal{D}_k \sim 1/k^2$ and $\mathcal{D}^{\text{in}}_k \sim 1/k^3$, respectively. These relations correspond to the case $\beta=2$ of the present model. Note that a correlation function of the Poincaré map of the mushroom billiard exhibits a power law decay $C(n) \sim 1/n$ [31], and this is consistent with the analytical result of the present paper. This relation between the map $\psi(x,y)$ and a billiard system is similar to that of the baker map and the Lorentz gas [10].

Since the map $\psi(x,y)$ is an elementary model of conservative systems and can be treated analytically to some extent, this system may be important for understanding the relationships between nonequilibrium phenomena, such as relaxation and transport, and underlying reversible dynamics; this is a fundamental problem in dynamical system theory and statistical mechanics [1,2].

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APPENDIX: ASYMPTOTIC BEHAVIORS

When $|z| \to 0$

In this appendix, we show the inequality

$$\left| \int_0^\infty ds \frac{s^{\beta-1} e^{-s}}{z - e^{-s}} \right| < C \left(\ln \frac{1}{|z|} \right)^\beta \tag{A1}$$

as $|z| \to 0$ and $\arg z \in [0, 2\pi]$ is fixed, where C is a positive constant. From this inequality, the Eqs. (49) and (50) can be derived. Let $z=x+iy$ in this subsection.

First, let us assume $\arg z \in [3\pi/4, 5\pi/4]$. Splitting the integral into two pieces, we have

$$\int_0^\infty ds \frac{s^{\beta-1}e^{-s}}{|z-e^{-s}|} \leq \int_0^{\ln(1/|x|)} ds s^{\beta-1} + \int_{\ln(1/|x|)}^\infty ds \frac{s^{\beta-1}e^{-s}}{|x|}. \tag{A2}$$

Apparently, the first term on the RHS of inequality (A2) has an upper bound $C_1(-\ln|z|)^\beta$ for some positive constant C_1. On the other hand, the second term has an upper bound $C_1'(-\ln|z|)^{\beta-1}$. This is obtained by using an asymptotic expansion of the incomplete γ function (see, e.g., Ref. [37]). Thus, in this case inequality (A1) holds.

Second, we consider the case for $\arg z \in [\pi/4, 3\pi/4]$ or $\arg z \in [5\pi/4, 7\pi/4]$. Using similar calculations, we have

$$\int_0^\infty ds \frac{s^{\beta-1}e^{-s}}{|z-e^{-s}|} \leq \left(\ln \frac{1}{|y|}\right)^\beta \int_0^1 ds \frac{1}{\sqrt{H(s)}} + \int_{\ln(1/|y|)}^\infty ds \frac{s^{\beta-1}e^{-s}}{|y|}, \tag{A3}$$

where we define $H(s)$ as $H(s) \equiv (x/|y|^s - 1)^2 + |y|^{2(1-s)}$. It is easy to check that $H(s) \geq 1/2$. Therefore, the first term has an upper bound $C_2(-\ln|z|)^\beta$. By using the asymptotic expansion of the incomplete γ function, we have an upper bound for the second term: $C_2'(-\ln|z|)^{\beta-1}$. Thus, this case also satisfies inequality (A1).

Finally, we consider the case $\arg z \in [0, \pi/4]$ or $\arg z \in [7\pi/4, 2\pi]$. We split the integral as

$$\int_0^\infty ds \frac{s^{\beta-1}e^{-s}}{z-e^{-s}} = \int_0^{x/2} dt \frac{G(t)}{z-t} + \int_0^{x/2} dt \left(\frac{G(x-t)}{t+iy} - \frac{G(x+t)}{t-iy}\right) + \int_{3x/2}^1 dt \frac{G(t)}{z-t}, \tag{A4}$$

where we define $G(t)$ as $G(t) \equiv \{\ln(1/t)\}^{\beta-1}$. By using similar techniques as above, it can be shown that the absolute values of the first and third terms on the RHS of Eq. (A4) have an upper bound $C_3(-\ln x)^\beta$. For the imaginary part of the second term, it is easy to derive an upper bound of its absolute value: $C_3'(-\ln x)^{\beta-1}$. On the other hand, for the real part, we have an upper bound $C_3'(-\ln x)^{\beta-2}$, which can be derived through integration by parts after changing the variables as $t' = t/x$. This completes the proof of inequality (A1).

When $|z| \to 1$

The property, Eq. (51), is also derived in the same way. We briefly comment about the derivation. Let us begin with an analytic continuation of $\Xi(z)$,

$$\Xi(z) = B_1^-(0)\Psi(z) + \sum_{j=0}^\infty \sum_{l=2}^\infty B_l^-(0) l^\beta \frac{\rho_j}{\Gamma(\beta)} \int_0^\infty ds \frac{s^{\beta-1}e^{-(j+l)s}}{z-e^{-s}}. \tag{A5}$$

We consider the second term. The absolute value of the second term has an upper bound

$$\text{const} \times \left| \sum_{j=0}^\infty \rho_j \int_0^\infty ds \frac{s^{\beta-1}e^{-(j+2)s}}{(z-e^{-s})(1-e^{-s})} \right|. \tag{A6}$$

Therefore, we show that for $k=1,2,\ldots$ and $1<\beta<2$,

$$\left| \int_0^1 ds \frac{s^{\beta-1}e^{-ks}}{(z-e^{-s})(1-e^{-s})} \right| < C'|1-z|^{\beta-2}, \tag{A7}$$

as $z \to 1$ and $\arg(z-1) \in [0, 2\pi)$ with a constant C'. Note that the integral in (A6) from 1 to ∞ is convergent. For $\beta \geq 2$, it can be analyzed in a similar way but the RHS of inequality (A7) should be changed to a constant ($\beta > 2$) or a logarithmic correction $-\ln|1-z|$ ($\beta = 2$). Let $z = 1 + x + iy$ in the following.

First, let us assume $\arg z \in [\pi/4, 3\pi/4]$ or $\arg z \in [5\pi/4, 7\pi/4]$; then, we have

$$\int_0^1 ds \frac{s^{\beta-1}e^{-ks}}{|(z-e^{-s})(1-e^{-s})|} < e \int_0^1 ds \frac{s^{\beta-2}}{|z-e^{-s}|}, \tag{A8}$$

where we have used $1 - e^{-s} \geq s/e$ for $s \in [0,1]$. The RHS of inequality (A8) can be estimated, by splitting the integral, as

$$\int_0^1 ds \frac{s^{\beta-2}}{|z-e^{-s}|} \leq \int_0^{2e|y|} ds \frac{s^{\beta-2}}{|y|} + 2e \int_{2e|y|}^1 ds s^{\beta-3}, \tag{A9}$$

where we have used $1 - e^{-s} \geq s/e$ again. It is obvious that the RHS of inequality (A9) is less than $C_4'|1-z|^{\beta-2}$. Therefore inequality (A7) is satisfied in this case.

Second, the case for $\arg z \in [0, \pi/4]$ or $\arg z \in [7\pi/4, 2\pi)$ can be analyzed in the same way as the first case (but in this case, split the integral in terms of x instead of y). Thus we omit the details.

Finally, when $\arg z \in [3\pi/4, 5\pi/4]$, we have

$$\int_0^1 ds \frac{s^{\beta-1}e^{-ks}}{(z-e^{-s})(1-e^{-s})} \tag{A10}$$

$$= \int_{1/e}^{1-3\epsilon/2} dt \frac{F(t)}{z-t} + \int_0^{\epsilon/2} dt \left(\frac{F(1-\epsilon-t)}{t+iy} - \frac{F(1-\epsilon+t)}{t-iy}\right) + \int_{1-\epsilon/2}^1 dt \frac{F(t)}{z-t}, \tag{A11}$$

where we define $\epsilon > 0$ as $\epsilon = |x|$ and $F(t)$ as $F(t) \equiv \{\ln(1/t)\}^{\beta-1} t^{k-1}/(1-t)$. The first and third terms of the RHS can be estimated in the same way as inequality (A9), and we obtain an upper bound for their absolute values as $C_5|1-z|^{\beta-2}$. For the imaginary part of the second term, we have easily a bound of its absolute value: $C_5'|1-z|^{\beta-2}$. On the other hand, for the real part, we also have a bound $C_5''|1-z|^{\beta-2}$ through integration by parts after changing the variables as $t' = t/\epsilon$. Thus we complete the proof of inequality (A7).

From inequality (A7), it can be shown that $\Psi(z) \to \Psi(1) < \infty$, as $z \to 1$. Thus the first term of Eq. (A5) converges as $z \to 1$. Consequently, we have Eq. (51).

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